Convergence of a sequence in a metric space:- Let $(E,d)$ be a metric space, A sequence $(x_n)$ in $E$ is said to converge to a point $x \in E$, if $\lim_{n\to\infty} d(x_n,x)=0$

i.e. the sequence $(d(x_n,x))$ converges to 0.

In other words $(x_n)$ converges to $x$ for every $\epsilon \geq 0$, $\exists$ a +ive integer $m$ such that

$d(x_n,x) < \epsilon$ for all $n \geq m$.

i.e., If $x_n \in S_\epsilon(x)$ for all $n \geq m$

So, $(x_n)$ converges to $x$ if for every $\epsilon > 0$ $\exists$ a +ive integer $m$ such that $x_n \in S_\epsilon(x)$ for all $n \geq m$.

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Q.No→ If a convergent sequence in a metric space has infinitely many distinct points, then show that its limit is a limit point of the set of points of the sequence.

Or, Q.No→ If a convergent sequence in a metric space has infinitely many distinct points, then its limit is an accumulation point (limit point) of the set of terms (points) of this sequence.

Proof:- Let $(x_n)$ be a sequence in a metric space $(E,d)$, having infinitely many distinct terms which converges to a point $x \in E$.

Let us suppose that $x$ is not an accumulation point of the set of terms of $(x_n)$. We deduce from it that the sequence $(x_n)$

has finitely many distinct points. Then there exists $r>0$ such that $S_r(x)$ doesn't contain any point of $(x_n)$ different of $x$. Also, $x_n \to x$. $S_r(x)$ will contain all but finitely many terms of $(x_n)$.

$\therefore x_n = x$ for all but finitely many $n$.

So $(x_n)$ has only finitely many distinct terms. This contradicts the assumption. Hence $x$ must be a limit point of the set of points of $(x_n)$.

Theorem

Q No → Let $(E,d)$ be a metric space. A subset $F$ of $E$ is closed iff for every sequence $(x_n)$ in $F$ converging to a point $x \in E$, we have $x \in F$.

In other words, $F$ is closed iff the limit of every convergent sequence in $F$ belongs to $F$.

or, Q No → In a metric space $(E,d)$ a set $F \subset E$ is closed iff for each sequence $(x_n)$ of points of $F$ that converges to a point $x \in E$, we have $x \in F$.

Proof :- Let $F$ be a closed set. Then $F^c$ is an open set. Let $(x_n)$ be a sequence in $F$ which converges to $x \in E$. If $x \in F^c$, then $F^c$ is an open set containing $x$. Since $x_n \to x$, then, there exists a ~~+ve~~ +ve integer $n_0$ such that $n \geq n_0 \Rightarrow x_n \in F^c$, which is a contradiction because $(x_n)$ is a sequence in $F$. Hence, $x \in F$.

Conversely, let for every sequence $(x_n)$ in $F$, with $x_n \to x \in E$, we have $x \in F$, we are required to show that F is closed.

For this, it is enough to show that $F^c$ is an open set, Let F is not an open set $p \in F^c$ be arbitrary.

Then for every nbd. $V_p$ of p, $V_p \cap F \neq \phi$

In particular, for every +ve integer $n_{V_p}$

$S_{1/n}(p) \cap F \neq \phi$.

i.e. for every +ve integer n | $x_n \in F$.

then,

$x_n \in S_{1/n}(p) \therefore d(x_n, p) < \frac{1}{n} \to 0$

As, $n \to \infty$, So $(x_n)$ Converges to p.

then $p \in F$, which is a Contradiction

Hence $F^c$ is an open set i.e.

F is a closed.

Theorem

QNo> Let C be subset C of metric space E and let $x \in E$. Then $x \in \overline{C}$ iff there exists a sequence in C converging to x.

Proof:- Let $x \in \overline{C}$. Then for every +ve integer n, $C \cap S_{1/n}(x) \neq \phi$, So for each +ve integer n, there exist $x_n \in C \cap S_{1/n}$ then, $x_n \in C$ for all n, i.e. $(x_n)$ is a sequence in C. Also, $x_n \in S_{1/n}(x) \Rightarrow d(x_n, x) < \frac{1}{n} \to 0$ as $n \to \infty \therefore x_n \to x$ as $n \to \infty$.

Conversely, let $(x_n)$ be a sequence

in $C$ Converging to $x$. We have to show that $x \in \overline{C}$. Let $\epsilon > 0$ be given. Since $x_n \to x$, there exist a +ve integer $n_0$ such that $x_n \in S_\epsilon(x)$ for all $n \geq n_0$. This implies that every open Sphere with Centre on $x$. Contains atleast one Point of $C$. So, $x \in \overline{C}$.